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Pion and vacuum properties in the nonlocal NJL model

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We formulate the nonlocal NJL model with a momentum dependent constituent quark mass and calculate pion light cone wave functions of twist 2 and 3. The leading twist wave function is not asymptotic and agrees well with the new CLEO data. Normalization conditions for the twist 3 wave functions are used to calculate the quark condensate. A prescription to calculate the gluon condensate is proposed. The numerical value of the gluon condensate nicely agrees with the phenomenological value, whereas the quark condensate is larger than the phenomenological value of $-(250\ \text{MeV})^3$. The relation between the k_T^2 moments and mixed condensates are used to estimate the mixed quark-gluon condensate of dimension 5.

I. INTRODUCTION

In this short note we shall describe a simple and tractable model for the pion light cone wave functions which is based on the instanton model of the QCD vacuum. Hadron light cone wave functions were theoretically introduced more than 20 years ago [1]- [5]. Recently the analysis of Ref. [6] based on the latest CLEO measurements [7] put some limits on the expansion coefficients of the axial-vector (AV) pion wave function in terms of the Gegenbauer polynomials. This analysis indicates that the pion wave function measured at $Q^2 = 1.5 - 9.2$ GeV^2 is neither asymptotic $\phi_{\rm as}^{AV}(u) = 6\,u(1-u)$ (with u being the fraction of the pion momentum carried by the quark) nor of the form proposed by Chernyak and Zhitnitsky in 1977 [8]: $\phi_{\rm CZ}^{AV}(u) = 30\,u(1-u)(1-2u)^2$. These two wave functions together with a typical prediction of the present model are shown in Fig.1a. In Fig. 1b we show the 95% and 68% confidence level contour plots in the $a_2 - a_4$ parameter space from the analysis of Schmedding and Yakovlev (Fig.6 in Ref. [6]) together with the values of a_2 and a_4 for $\phi_{\rm as}^{AV}$, $\phi_{\rm CZ}^{AV}$ and various parameters of the present model.

The instantanton model, after integrating out gluons and performing the bosonization, reduces to a simple Nambu-Jona Lasinio type model where the quarks interact *nonlocally* with an external meson field U [9,10]:

$$S_I = \int \frac{d^4k d^4l}{(2\pi)^8} \bar{\psi}(k)\sqrt{M(k)}U^{\gamma_5}(k-l)\sqrt{M(l)}\psi(l) \tag{1}$$

and U^{γ_5} can be expanded in terms of the pion fields:

$$U^{\gamma_5} = 1 + \frac{i}{F_\pi}\gamma^5\tau^A\pi^A - \frac{1}{2F_\pi^2}\pi^A\pi^A + \dots \tag{2}$$

Here $F_\pi = 93$ MeV and $M(k) = MF^2(k)$ is a momentum dependent constituent quark mass which also plays a role of the pion-quark coupling. Let us note that in the instanton model both quark and gluon condensation occur at the same scale μ_0 which is associated with the average instanton size $1/\overline{\rho} = 600$ MeV.

In principle $F(k)$ has been calculated in the instanton model in the Euclidean space time. Here, following Refs. [11,12], we will perform calculations directly in the Minkowski space. To this end we shall choose a simple pole formula [12]

$$F(k) = \left(-\frac{\Lambda^2}{k^2 - \Lambda^2 + i\epsilon}\right)^n \tag{3}$$

which for $n \sim 2-3$ and for $k^2 < 0$ reproduces the k dependence obtained from the instantons reasonably well [12] (see Fig. 2a). Here $M = M(0)$ is a model parameter which we choose to be of the order of 350 MeV.

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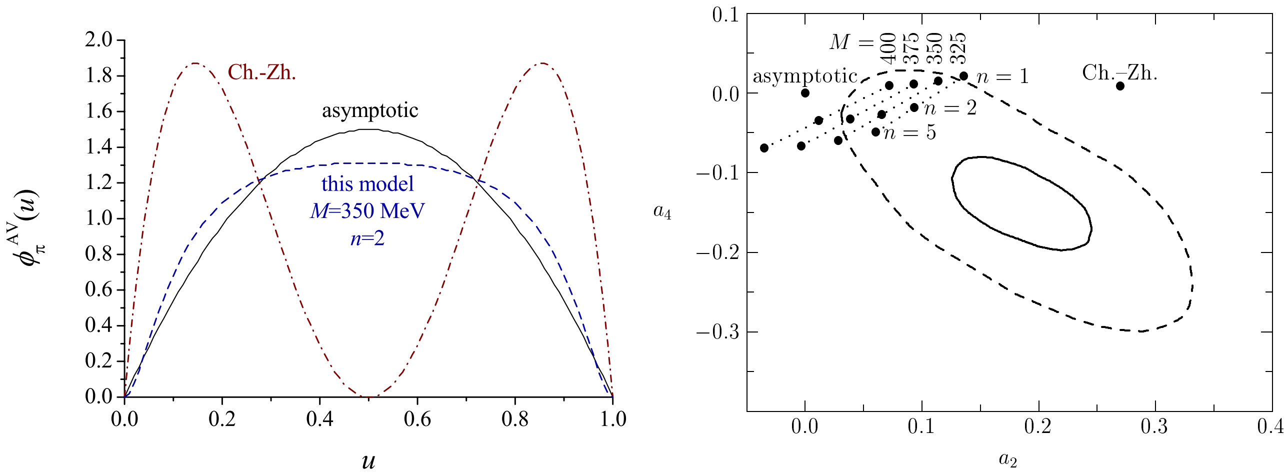

FIG. 1. Left: Asymptotic and Chernyak-Zhytnitsky leading twist pion wave functions together with a typical wave function from the present model. Right: The parameter space (a_2, a_4) of Ref.[6]. Black dots represent different model predictions, solid contour corresponds to 68% confidence level, whereas the dashed one to 95%

As we shall see, the model is technically very simple and allows to calculate pion wave functions (not only the axial-vector, but also the pseudo-scalar (PS) and the pseudo-tensor (PT) ones) analytically up to a numerical solution of a certain algebraical equation of the order $4n+1$. Given this simplicity it is of importance to perform various tests in order to gain confidence in the model as well as to find its limitations. In this paper we provide 4 kinds of tests.

First we calculate the leading twist pion wave function and compare with the existing data. Next we calculate the non-leading twist wave functions, which are normalized to the quark condensate. This allows us to calculate $\langle \bar{q}q \rangle$.

It is important to note that in our approach we calculate not only the u dependence but also the dependence on the transverse momentum k_T:

$$\phi_\pi(u) = \int_0^\infty dk_T^2\, \psi_\pi(u, k_T^2), \qquad \tilde{\phi}_\pi(k_T^2) = \int_0^1 du\, \psi_\pi(u, k_T^2). \tag{4}$$

By calculating k_T^2 moments we get the mixed condensate of dimension 5.

Another advantage of our method is that the analytical expression for the quark condensate is given in terms of a Minkowskian integral which in a limit of a constant $M(k)$ and $k^2 \to -k_E^2$ reduces to the well known Euclidean form. By comparing the two expressions one can by inspection guess a continuation prescription which allows to rewrite certain Euclidean integrals as the Minkowskian ones. We use this in some respect *ad hoc* prescription to calculate the gluon condensate $\langle \alpha/\pi\, GG \rangle$, which provides another test of our approach.

II. PION WAVE FUNCTIONS IN THE NONLOCAL QUARK MODEL

We shall be dealing with the leading twist axial-vector (AV), twist 3 pseudo-scalar (PS) and pseudo-tensor (PT) wave functions defined as follows [13,14]:

$$\phi_\pi^{AV}(u) = \frac{1}{i\sqrt{2}F_\pi} \int_{-\infty}^{\infty} \frac{d\tau}{\pi} e^{-i\tau(2u-1)(nP)} \left\langle 0 \right| \bar{\psi}(n\tau) \not{n} \gamma_5 \psi\left(-n\tau\right) \left| \pi^+(P) \right\rangle, \tag{5}$$

$$\phi_\pi^{PS}(u) = -(nP) \frac{F_\pi}{\sqrt{2}\left\langle \bar{q}q \right\rangle} \int_{-\infty}^{\infty} \frac{d\tau}{\pi} e^{-i\tau(2u-1)(nP)} \left\langle 0 \right| \bar{\psi}(n\tau) i\gamma_5 \psi\left(-n\tau\right) \left| \pi^+(P) \right\rangle,$$

$$\phi_\pi^{PT}(u) = \frac{-6F_\pi}{\sqrt{2}\left\langle \bar{q}q \right\rangle} \int_0^u dw \int_{-\infty}^{\infty} \frac{d\tau}{\pi} e^{-i\tau(2w-1)(nP)} n^\alpha P^\beta \left\langle 0 \right| \bar{\psi}(n\tau) \sigma_{\alpha\beta} \gamma_5 \psi\left(-n\tau\right) \left| \pi^+(P) \right\rangle.$$

where we have chosen $n = (1, 0, 0, -1)$ as a light-cone vector parallel to $z_\mu = \tau n_\mu$ and $\tilde{n} = (1, 0, 0, 1)$ parallel to P_μ. All three wave functions are normalized to 1. The normalization condition for ϕ_π^{PS} and yield ϕ_π^{PS} therefore the expression for $\langle \bar{q}q \rangle$, whereas normalization of ϕ_π^{AV} is used to fix the model parameter Λ for given M and n.

Technically speaking all three wave functions (5) are given in terms of a loop integral with a momentum dependent quark mass $M(k)$, which also acts as a quark-pion coupling. In order to calculate the loop integral we have to find zeros of the propagators which are generically of the form

$$k^2 - M^2 \left[\frac{\Lambda^2}{k^2 - \Lambda^2 + i\epsilon} \right]^{4n} + i\epsilon = 0. \tag{6}$$

Equation (6) can be conveniently rewritten as:

$$z^{4n+1} + z^{4n} - \mu^2 = 0 \tag{7}$$

where $z = k^2/\Lambda^2 - 1 + i\epsilon$ and $\mu^2 = M^2/\Lambda^2$. In the light cone parametrization $d^4k = 1/2\, dk^+ dk^- d^2\vec{k}_T$ where $k^\mu = k^+ \tilde{n}^\mu/2 + k^- n^\mu/2 + k_T^\mu$. Since $k^+ = uP^+$ is fixed, equation (6) should be understood as an equation for k^-. Generally, equation (7) has $4n+1$ complex solutions which in the following will be denoted as z_i. These solutions depend on the specific value of μ^2 and have to be calculated numerically.

Here one faces immediately the problem how to choose the integration contour in the complex k^- plane. The prescription is very simple and has been at length discussed in Ref. [12]. As a result the dk^- integrals yield real wave functions which vanish for u outside the region $0 < u < 1$. Moreover for $\Lambda \to \infty$, i.e. for a constant $M(k)$, this prescription reduces in a continuous way to the standard one of Feynman.

With this prescription the calculations are rather straightforward and we obtain:

$$\psi_\pi^{AV}(u, k_T^2) = \frac{N_c}{(2\pi)^2} \frac{M^2}{\Lambda^2 F_\pi^2} \sum_{i,k=1}^{4n+1} f_i f_k \frac{u z_i^n z_k^{3n} + (1-u) z_i^{3n} z_k^n}{t + 1 + u z_i + (1-u) z_k}, \tag{8}$$

$$\psi_\pi^{PS}(u\,, k_T^2) = \frac{N_c}{(2\pi)^2} \frac{M}{\langle \bar{q}q \rangle} \sum_{i,k=1}^{4n+1} f_i f_k \frac{z_i^{3n} z_k^{3n} (1 + \frac{z_i + z_k}{2}) - \mu^2 z_i^n z_k^n}{t + 1 + u z_i + (1-u) z_k}, \tag{9}$$

$$\psi_\pi^{PT}(u\,, k_T^2) = \frac{3N_c}{(2\pi)^2} \frac{M\Lambda^2}{\langle \bar{q}q \rangle} \sum_{i,k=1}^{4n+1} f_i f_k z_i^{3n} z_k^{3n} \ln(1 + t + u z_i + (1-u) z_k). \tag{10}$$

(where $t = k_T^2/\Lambda^2$). Factors f_i obey the following properties:

$$f_i = \prod_{\substack{k=1 \\ k \neq i}}^{4n+1} \frac{1}{z_i - z_k}, \quad \sum_{i=1}^{4n+1} z_i^m f_i = \begin{cases} 0 & \text{for} \quad m < 4n \\ 1 & \text{for} \quad m = 4n \end{cases} \tag{11}$$

which are crucial for the convergence of the dt integrals.

It is now straightforward to perform either the dt integration in order to get ϕ_π, or the du integration to get the k_T -dependent functions $\tilde{\phi}_\pi$.

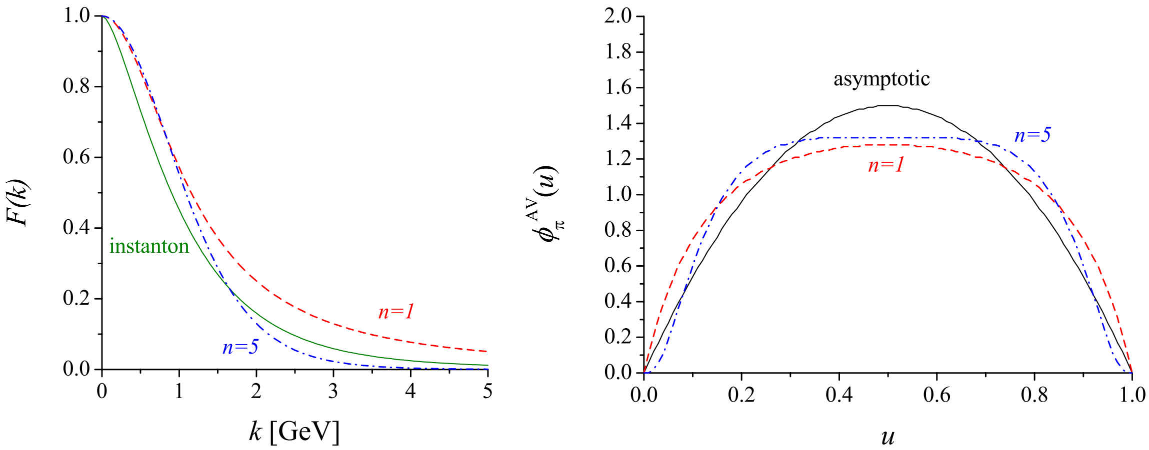

FIG. 2. Left: $F(k)$ for Euclidean momentum $k^2 < 0$, for $n = 1$ (dashed), 5 (dashed-dotted) and for the instanton model (solid). Right: Axial-vector pion wave function for $M = 350$ MeV and for $n = 1$ (dashed) and 5 (dashed-dotted) together with the asymptotic one (solid)

In order to study the model dependence on the choice of M and n we have calculated pion wave functions for $M = 325 - 400$ MeV and $n = 1 - 5$. The cutoff parameter Λ was adjusted by imposing the normalization condition on ϕ_π^{AV}. In fact, as discussed in Ref. [12], the leading twist pion wave function $\phi_\pi^{AV}(u)$ does not change any more if we increase n above 5. On the other hand for $n > 5$ the cutoff function (3), if continued to the Euclidean metric, starts to deviate significantly from the one obtained in the instanton model. Therefore we have chosen to work with $n_{\rm max} = 5$. In Figs. 2b and 3 we have plotted ϕ_π^{AV}, ϕ_π^{PS} and ϕ_π^{PT} for $M = 350$ MeV and $n = 1, 5$.

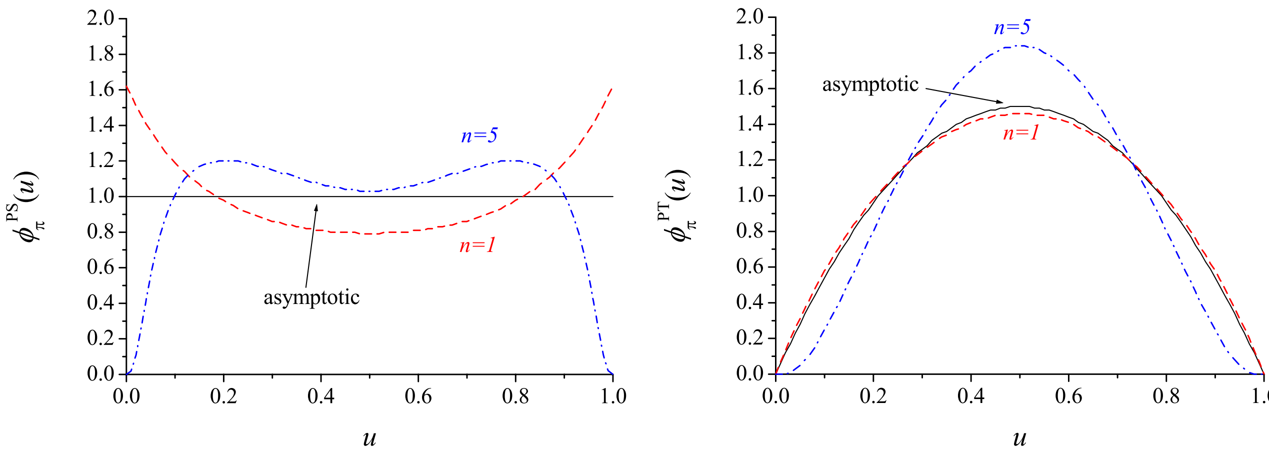

FIG. 3. Pseudo scalar (left panel) and pseudo-tensor (right panel) pion wave function for $M = 350$ MeV and for $n = 1$ (dashed) and 5 (dashed-dotted) together with the asymptotic one (solid)

Let us shortly summarize our findings. The axial-vector wave function, ϕ_π^{AV}, vanishes at the end points as u^n (or $(1-u)^n$) and shows a plateau around $u = 0.5$ with a small dip for $n = 5$. It differs from the asymptotic wave function $\phi_{\rm as}^{AV}$ and, as seen from Fig. 1b, the best agreement with the recent analysis of the CLEO data is obtained for $M = 325$ MeV and $n = 2 - 5$ or $M = 350$ MeV and $n = 2$. The fact that the true pion distribution amplitude may be broader than the asymptotic one has been already pointed out in Ref. [15]. Such a behavior was found then in Ref. [16] where not only the nonlocality (within the sum rules approach) but also the radiative corrections have been taken into account.

The pseudo-scalar pion wave function, ϕ_π^{PS}, was calculated within the QCD sum rules in Refs. [13,14]. It had a $u-$ shape and did not vanish at the end points. In our case ϕ_π^{PS} vanishes at the end points for $n > 1$. Indeed

$$\phi_\pi^{PS}(1) \sim \sum_i f_i \ln(1 + u z_i) \sum_k f_k \left[z_i^{3n} z_k^{3n} + \frac{1}{2} z_i^{3n+1} z_k^{3n} + \frac{1}{2} z_i^{3n} z_k^{3n+1} - \mu^2 z_i^n z_k^n \right] \tag{12}$$

(and similarly for $u = 0$) which is equal to 0 due to the property (11) except for $n = 1$ where $3n + 1 = 4n$. Interestingly, the vanishing of ϕ_π^{PS} for $u = 0, 1$ is correlated with the nonconvexity of ϕ_π^{AV} at the end points, which, as stated above, behaves like u^n (or $(1-u)^n$) for $u \to 0$ (or 1). In any case ϕ_π^{PS} differs from its asymptotic form $\phi_{\rm as}^{PS} \equiv 1$.

Both pseudo-scalar and pseudo-tensor wave functions show stronger n dependence than ϕ_π^{AV}. For $n = 1$ ϕ_π^{PT} coincides with the asymptotic expression $\phi_{\rm as}^{PT} = \phi_{\rm as}^{AV}$, while for $n = 5$ its is depleted at the end points and peaked in the center.

IV. CONDENSATES

Since the model parameters are fixed by the normalization of the axial-vector wave function we could use the normalization condition for ϕ_π^{PS}or ϕ_π^{PT} to calculate the quark condensate. The results are presented in Table 1. We see that the quark condensates obtained from the two normalization conditions do not coincide. In fact, for almost all model parameters considered, we find

$$\sqrt[3]{\langle \bar{q}q \rangle_{PS} \,/\, \langle \bar{q}q \rangle_{PT}} \simeq 0.9. \tag{13}$$

In absolute values the quark condensate calculated within our model overshoots the phenomenological value of $-(250 \text{ MeV})^3$. This is mostly due to the rather poor convergence of the dk_T^2 integrals (4). Indeed for large k_T^2:

$$\tilde{\phi}_\pi^{AV}(k_T^2) \sim \left(\frac{1}{k_T^2}\right)^{4n+1}, \; \tilde{\phi}_\pi^{PS}(k_T^2) \sim \left(\frac{1}{k_T^2}\right)^{2n}, \; \tilde{\phi}_\pi^{PT}(k_T^2) \sim \left(\frac{1}{k_T^2}\right)^{2n}. \tag{14}$$

This is also the reason of rather strong n dependence of $\phi_\pi^{PS}$and ϕ_π^{PT}.

The Euclidean formula for the gluon condensate in the instanton model of the QCD vacuum reads [9]:

$$\left\langle \frac{\alpha}{\pi} GG \right\rangle = 32 N_c \int \frac{d^4 k_E}{(2\pi)^4} \frac{M^2(k_E)}{k_E^2 + M^2(k_E)}. \tag{15}$$

Apart from the numerical factor in front it differs from $\langle \bar{q}q \rangle$ by an additional power of $M(k_E)$ in the numerator. In Ref. [12] we have suggested the continuation prescription of (15) to the Minkowski metric with the result

$$\left\langle \frac{\alpha}{\pi} GG \right\rangle = -\frac{8 N_c M^2 \Lambda^2}{(2\pi)^2} \int du \; dt \sum_{i,k} f_i f_k \frac{z_i^{2n} z_k^{2n} (1 + \frac{z_i + z_k}{2}) - \mu^2}{t + 1 + u z_i + (1-u) z_k}. \tag{16}$$

Numerical result (Table 1) depends very weakly on n and is compatible with the phenomenological value [17]: $\langle \alpha/\pi \; GG \rangle = (393^{+29}_{-38} \text{ MeV})^4$.

TABLE I. Condensates for $M = 350$ MeV

n	Λ	$\left\langle \frac{\alpha}{\pi} GG \right\rangle$	$\langle \bar{q}q \rangle_{PS}$	$\langle \bar{q}q \rangle_{PT}$	$\langle ig\,\bar{q}\,\sigma \cdot G\,q \rangle_{AV}$
1	1156 MeV	$(399 \text{ MeV})^4$	$-(318 \text{ MeV})^3$	$-(357 \text{ MeV})^3$	$-(553 \text{ MeV})^5$
5	2819 MeV	$(389 \text{ MeV})^4$	$-(271 \text{ MeV})^3$	$-(301 \text{ MeV})^3$	$-(475 \text{ MeV})^5$

Soft pion theorems provide link between dynamical objects like the light cone wave functions [1–4] and static properties of the physical vacuum [5,18]. It has been shown in Refs. [5,18] that moments of $\tilde{\phi}_\pi(k_T^2)$ are given in terms of the mixed quark-gluon condensates

$$\left\langle k_T^2 \right\rangle_{AV} = \frac{5}{36}\frac{\langle ig\,\bar{q}\,\sigma\cdot G\,q\rangle}{\langle \bar{q}q\rangle},\ \left\langle k_T^2 \right\rangle_{PS} = \frac{1}{4}\frac{\langle ig\,\bar{q}\,\sigma\cdot G\,q\rangle}{\langle \bar{q}q\rangle}. \tag{17}$$

Here $G^a_{\mu\nu}$ is a gluon field strength and $\sigma\cdot G = \sigma_{\mu\nu}G^{\mu\nu}$, $G_{\mu\nu} = \lambda^a/2\ G^a_{\mu\nu}$.

Unfortunately the ratio $\left\langle k_T^2 \right\rangle_{AV} / \left\langle k_T^2 \right\rangle_{PS} \sim 5/9$ which follow from (17) is not reproduced within our approach[1] due to the slow convergence of the dk_T^2 integration in the case of $\left\langle k_T^2 \right\rangle_{PS}$. In order to estimate the value of the mixed condensate of dimension 5, $\langle ig\,\bar{q}\,\sigma\cdot G\,q\rangle$, we choose therefore the first equation of Eqs.(17). Interestingly, for the parameters which are closest to the original instanton model, $M = 350$ MeV and $n = 2$, we get $-(493$ MeV$)^5$ in perfect agreement with the direct calculation of $\langle ig\,\bar{q}\,\sigma\cdot G\,q\rangle$ in the instanton model [19] which gives $-(490$ MeV$)^5$.

V. SUMMARY AND OUTLOOK

The nonlocal NJL model with the momentum dependent constituent quark mass has been applied to calculate pion light cone wave functions [12]. It gives a satisfactory description of the leading twist AV wave function, whereas for the twist 3 wave functions we find a somewhat larger sensitivity to the model parameters.

Present prescription can be easily extended to describe kaon wave functions with an explicit symmetry breaking due to the non zero current strange quark mass. Also two meson generalized parton distributions both for pions and kaons can be easily calculated. By crossing symmetry one can also apply our method to calculate the skewed distributions and structure functions [22,23].

On the theoretical side one has to investigate more closely the PCAC relation within the present approach. It is known that the properly defined currents should include additional terms with respect to those considered here [20,21]. Although these new terms are not unique and suppressed by the instanton packing fraction, their influence on our results should be investigated.

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